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Dilatonic Black Holes Time Stability

O.S. Khovanskaya[1]

Sternberg Astronomical Institute, Universitetskii Prospekt, 13, Moscow 119992, RUSSIA

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The stability under small time perturbations of the dilatonic black hole solution near the determinant curvature singularity is proved. This fact gives the additional arguments that the investigated topological configuration can realise in nature. In the frames of this model primordial black hole remnants are examined as time stable objects, which can form an significant part of a dark matter in the Universe.

One rather important result of the string gravity is a restriction upon the minimal black hole mass [1], [2], [3] in frames of the model with the 4D effective string action, containing graviton, dilaton and higher order cur vature corrections. Let me consider the action in the following form [1]:

$$\begin{aligned} S &= \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[m_{Pl}^2\Big(-R+2\partial_\mu\phi\partial^\mu\phi\Big) \\ &+ \lambda\Big(e^{-2\phi}(R_{ijkl}R^{ijkl}-4R_{ij}R^{ij} \\ &+ R^2)\Big)+\ldots\Big], \end{aligned} \quad (1)$$

where m_{Pl} is Plank mass, λ is the string coupling constant, ϕ is the dilatonic scalar field. (The system of units where $\hbar = c = G = 1$ and $m_{Pl} = 1$ is used. The community of the considered problem is not restricted by choosing $\lambda = 1$).

The restriction upon the minimal black hole mass is absent in the Schwarzschild solution of Einstein's classical gravity. This "mathematical" result can be put into practice in modern cosmology to study the remnants of primordial black holes [4], [5]. Such remnants can represent the final stage of Hawking evaporation of primordial black holes, formed in the early Universe, and are considered as dark matter candidates [6], [7].

The search of the exact solutions or at least the numerical ones in the offered model (1) with the metric depended on two parameters, the radial co-ordinate and the time, is known to be very difficult [1], [2], [3]. N evertheless one can receive the general properties of the time-evolution of such solutions by study its stability about time-parameter in all particular points.

The black hole solution in the frames of considered model has only two particular points. The first one is the usual coordinate singularity r_h, which represents the event regular horizon of a black hole. The second particular point is the determinant c urvature singularity, which has the infinite derivatives of the metric functions [2].

Primordial black hole stability on the event horizon was investigated in [8], [9].

It was obtained (P. Kanti [8] at al.) that the dilatonic black holes are stable near the black hole regular horizon r_h under linear time-dependent perturbations, which depend on only one radial parameter. The metric parameterisation was

$$ds^2 = e^{\Gamma(r,t)}dt^2 - e^{\Lambda(r,t)}dr^2 - r^2(d\,\theta^2 + \sin^2\,\theta d\,\psi^2)$$

and the asymptotic forms of metric components and dilaton field near the regular horizon $r \approx r_h$ were

$$\begin{aligned} e^{-\Lambda(r)} &= \lambda_1\,(r-r_h)+\lambda_2\,(r-r_h)^2+\ldots, \\ e^{\Gamma(r)} &= \gamma_1\,(r-r_h)+\gamma_2\,(r-r_h)^2+\ldots, \\ \phi(r) &= \phi_h+\phi_h'\,(r-r_h)+\phi_h''\,(r-r_h)^2+\ldots, \end{aligned}$$

where $\lambda_1 = 2/(\lambda e^{\phi_h}\phi_h'/g^2 + 2r_h)$.

P. Kanti at al. considered perturbing equations by time-dependent linear perturbations of the form:

$$\begin{aligned} \Gamma(r,t) &= \Gamma(r)+\delta\,\Gamma(r,t)=\Gamma(r)+\delta\,\Gamma(r)\,e^{i\,\omega\,t}, \\ \Lambda(r,t) &= \Lambda(r)+\delta\,\Lambda(r,t)=\Lambda(r)+\delta\,\Lambda(r)\,e^{i\,\omega\,t}, \\ \phi(r,t) &= \phi(r)+\delta\,\phi(r,t)=\phi(r)+\delta\,\phi(r)\,e^{i\,\omega\,t}, \end{aligned}$$

where the variations $\delta\,\Gamma(r,t), \delta\,\Lambda(r,t)$ and $\delta\,\phi(r,t)$ were assumed to be small. The stability problem was reduced to one-dimensional Schrodinger problem [8].

The regular horizon stability was investigated also in the paper by T.Torii and K.-i.Maeda [9]. They used the catastrophe theory and compared it with linear perturbation analysis. Generally the catastrophe theory is a mathematical tool to investigate a variety of some physical states, T.Torii and K.-i.Maeda showed this method is also applicable to the stability analysis of various types of non-Abelian black holes [10].

It is necessary to make more careful stability analysis under the horizon with the help of suitable choice of

[1] e-mail: khovansk@xray.sai.msu.ru

asymptotic forms that approximate the metric functions. In this work I investigate the dilatonic black hole stability near the determinant curvat ure singularity $r = r_s$ [2]. After the definition of a rest point, this problem can be reduced to a one-dimensional Schrodinger problem under the variation of the field equation. One can prove that the small time perturbations do not increase. S o in that case the solution of the dilatonic black hole is stable near r_s.

To investigate the stability under time perturbations of the dilatonic black hole near the singularity $r \approx r_s$, I use the model (1) with the following non-static, asymptotically flat spherically symmetric metric [1]:

$$ds^2 = \Delta dt^2 - \frac{\sigma^2}{\Delta} dr^2 - r^2 (d\,\theta^2 + \sin^2\,\theta d\,\psi^2), \quad (2)$$

where $\Delta = \Delta(r,t)$, $\sigma = \sigma(r,t)$, $\phi = \phi(r,t)$.

According to Kanti's method [8] I produce the metric functions in the form:

$$\Delta(r,t) = \Delta(r) + \delta\,\Delta(r,t) = \Delta(r) + \delta\,\Delta(r)\, e^{i\,\omega\,t}, \quad (3)$$

$$\sigma(r,t) = \sigma(r) + \delta\,\sigma(r,t) = \sigma(r) + \delta\,\sigma(r)\, e^{i\,\omega\,t}, \quad (4)$$

$$\phi(r,t) = \phi(r) + \delta\,\phi(r,t) = \phi(r) + \delta\,\phi(r)\, e^{i\,\omega\,t}, \quad (5)$$

where the variations $\delta\,\Delta(r,t), \delta\,\sigma(r,t)$ and $\delta\,\phi(r,t)$ are assumed to be small [1].

Thus for such definition of variables r and t in (3)-(5) the asymptotic forms of metric functions near the singularity $r \approx r_s$ depend on only radial coordinate r and do not depend on time:

$$\Delta(r) = \delta_0 + \frac{\phi_2\,(r - r_s)}{\theta} + \delta_3\,(r - r_s)^{3/2} + \ldots, \quad (6)$$

$$\sigma(r) = \sigma_0 + \sigma_2\,\sqrt{r - r_s} + \ldots, \quad (7)$$

$$\phi(r) = \phi_0 + \phi_2\,(r - r_s) + \phi_3\,(r - r_s)^{3/2} + \ldots, \quad (8)$$

where

$$\phi_0 = \frac{3}{4}\ln(2) - \frac{1}{2}\ln(-\eta) + \ln(1-\eta) - \ln(r_s),$$

$$\phi_2 = \frac{1}{2}\,\frac{(\eta - 1)\sqrt{2}}{\eta\, r_s},$$

$$\begin{aligned}\phi_3 \;&=\; -\frac{1}{96 r_s\,\eta^4}\,\sigma_2\,(\eta-1)^2\sqrt{2}\,(-8\,\eta + 8\,\eta^3 \\ &-\; 20\,\eta^2 + 4\,\eta^4 - 4\cdot 2^{3/4}\,\eta^2 + 3\cdot 2^{3/4}\,\eta^4 + 2^{3/4}),\end{aligned}$$

$$\sigma_0 = 4\,\frac{\eta}{1-\eta^2},$$

$$\delta_0 = -16\,\frac{\eta^2}{(1-\eta^2)^2},$$

$$\delta_3 = -\frac{16}{3}\,\frac{\sigma_2\,(\eta-1)}{r_s\,(1+\eta)^2}$$

and

$$\theta = \frac{1}{32}\,\frac{\sqrt{2}\,(1-\eta^2)^2}{\eta^2}.$$

It is possible to note that $\eta \in (-1, 0)$. We have three free parameters η, σ_2 and r_s [2].

The exact field equations for (1)-(2) which depend on r and t are in Appendix. Transforming (14)-(17), one can obtain the autonomous (over t) equation system of the first order in the following form:

$$\begin{cases} \dot{\Delta} &= \alpha \\ \dot{\sigma} &= \beta \\ \dot{\phi} &= \gamma \\ \dot{\alpha} &= -G \\ \dot{\beta} &= 2G/\Lambda \\ \dot{\gamma} &= F \end{cases} \quad (9)$$

where G, F are functions of $r, \Delta, \Delta', \Delta'', \sigma, \sigma', \phi, \phi', \phi''$ and α, β, γ are additional variables.

The function $\Lambda = -2\Delta/\sigma$. Using the asimptotic form (6)-(7) one can obtained that $\Lambda = 8\eta/(1-\eta^2)$ for $r = r_s$. $\Lambda \neq 0, \Lambda \neq \infty$ near the singularity $r \approx r_s$ because the dilaton function $\phi(r)$ (8) is limited in this region [2].

Let me examine the equilibrium states of the autonomous system (9).

Let the point $(\Delta^*,\, \sigma^*,\, \phi^*,\, \alpha^*,\, \beta^*,\, \gamma^*)$ is some rest point of the system (9), that is for

$$f_i \in \{\alpha,\, \beta,\, \gamma,\, -G,\, 2G/\Lambda,\, F\},$$

$$i = 1, 2, \ldots, 6$$

the condition

$$f_i(\Delta^*,\, \sigma^*,\, \phi^*\, \alpha^*,\, \beta^*,\, \gamma^*) = 0$$

is executed. The trivial "equilibrium-like" solution which corresponds to the given rest point is asymptotically stable if the first order system is stable. It occurs when the all roots s of the characteristic equation

$$det\Big[\Big(\frac{df_i}{dy_k}\Big)|_{rest\ point} - s\cdot\delta^i_k\Big] \;=\; 0 \quad (10)$$

where $y_k \in \{\Delta^*,\, \sigma^*,\, \phi^*,\, \alpha^*,\, \beta^*,\, \gamma^*\}$ $(k = 1, 2, \ldots, 6)$ have the negative real parts.

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[1] In this work I only study the small time perturbations of the diagonal metric components because in general I investigate the case of spherically symmetric metric. I'm not interesting in some rotating effects, which can appear because of non-zero non-diagonal metric components.

[2] In this case it is conveniently to choose η, σ_2 and r_s as free. There are direct three free parameters which can be reduced to the usual free parameters: the black hole mass, the dilaton charge and the dilaton value at infinity [2].

It is possible to find the rest point of the system (9) using the asymptotic forms (6)-(8) near the singularity $r \approx r_s$ from the condition:

$$\left\{ \begin{array}{lcl} F|_{rest\,point} & = & 0 \\ G|_{rest\,point} & = & 0 \end{array} \right. \tag{11}$$

In the rest point $\alpha = \beta = \gamma \equiv 0$ and $\eta = \eta^* = C_1,\ \sigma_2 = \sigma_2(r_s) = C_2/r_s^{1/2} < 0$, where C_1 and C_2 are the number coefficients.

By solving (10) one can receive the pure imaginary roots s. Thus some additional investigations for determine the stability are required.

Using the method of linear stability one can rewrite the field equations (14)-(17) with the variations:

$$\begin{array}{lcl} 0 & = & A_{i\,1}\delta\,\ddot{\phi} + A_{i\,2}\delta\,\dot{\phi} + A_{i\,3}\delta\,\phi + A_{i\,4}\delta\,\phi' \\ & + & A_{i\,5}\delta\,\phi'' + B_{i\,1}\delta\,\ddot{\Delta} + B_{i\,2}\delta\,\dot{\Delta} + B_{i\,3}\delta\,\Delta \\ & + & B_{i\,4}\delta\,\Delta' + B_{i\,5}\delta\,\Delta'' + C_{i\,1}\delta\,\ddot{\sigma} + C_{i\,2}\delta\,\dot{\sigma} \\ & + & C_{i\,3}\delta\,\sigma + C_{i\,4}\delta\,\sigma' + C_{i\,5}\delta\,\sigma'', \end{array} \tag{12}$$

where $i = 1..4$. Nonzero coefficients are in Appendix.

In the vicinity of finding from (11) rest point $O^+(0,(r-r_s)),\ (r-r_s) \sim 10^{-3}$ it is possible to evaluate the coefficients in front of the variations in (12) using the asymptotic forms (6)-(8). The simplified equation from (12) becomes the following:

$$A(\delta\phi)'' + B(\delta\phi)' + C(\delta\phi) \quad = \quad \omega^2(\delta\phi), \tag{13}$$

where A, B and C near the singularity $r \approx r_s$ are:

$$\begin{array}{lcl} A & = & \dfrac{a_1}{r_s} + \dfrac{a_2\,(r-r_s)^{1/2}}{r_s^{3/2}} + O(r-r_s), \\ B & = & \dfrac{b_1}{r_s^2} + \dfrac{b_2\,(r-r_s)^{-1/2}}{r_s^{5/2}} + O((r-r_s)^{1/2}), \\ C & = & \dfrac{c_1}{r_s^3} + \dfrac{c_2\,(r-r_s)^{-1/2}}{r_s^{5/2}} + O((r-r_s)^{1/2}), \end{array}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ depend on σ_2 and η. For $r = r_s$ these coefficients are number constants.

The eigenvalue ω^2 in (13) has only positive values. Thus, the small time perturbations do not increase (they can oscillate). So in that case the solution is stable in the vicinity of finding rest point $(O^+(0,(r-r_s))$.

Taking into account both results: the stability of the solution under time perturbations at the regular event horizon r_h [8] and at the determinant curvature singularity r_s, it is possible to conclude that the solution of dilatonic black ho le is stable in all particular points. In application to cosmology this fact can confirm the existence of remnants of primordial black holes, which are stable during time evolution.

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Appendix

The exact field equations for (1)-(2) which depend on r and t are the following unwieldy form:

$$\begin{array}{lcl} 0 & = & 2\,e^{2\,\phi}\,r^2\,\phi'^2\,\sigma^3\,\Delta^2 - 8\,\phi''\,\sigma^3\,\Delta^2 \\ & - & 2\,e^{2\,\phi}\,r\,\sigma'\,\sigma^2\,\Delta^2 + 16\,\phi'^2\,\sigma^3\,\Delta^2 \\ & + & 8\,\phi'\,\sigma'\,\sigma^2\,\Delta^2 + 2\,\sigma^5\,e^{2\,\phi}\,r^2\,\dot{\phi}^2 \\ & + & 8\,\dot{\phi}\,\dot{\sigma}\,\sigma^4 - 8\,\dot{\phi}\,\dot{\sigma}\,\sigma^2\,\Delta \\ & - & 16\,\dot{\phi}^2\,\sigma^3\,\Delta + 16\,\sigma^5\,\dot{\phi}^2 \\ & - & 8\,\sigma^5\,\ddot{\phi} + 8\,\ddot{\phi}\,\sigma^3\,\Delta \\ & - & 16\,\Delta^3\,\phi'^2\,\sigma - 24\,\Delta^3\,\phi'\,\sigma' + 8\,\Delta^3\,\phi''\,\sigma, \end{array} \tag{14}$$

$$\begin{array}{lcl} 0 & = & -2\,e^{2\,\phi}\,\sigma^4\,\Delta^2 + 2\,e^{2\,\phi}\,r\,\Delta'\,\sigma^2\,\Delta^2 \\ & - & 2\,\Delta^3\,e^{2\,\phi}\,r^2\,\phi'^2\,\sigma^2 + 24\,\Delta^3\,\phi'\,\Delta' + 8\,\dot{\phi}\,\dot{\Delta}\,\sigma^2\,\Delta \\ & - & 8\,\phi'\,\Delta'\,\sigma^2\,\Delta^2 + 2\,\Delta^3\,e^{2\,\phi}\,\sigma^2 \\ & - & 16\,\ddot{\phi}\,\sigma\,\Delta^2 + 32\,\dot{\phi}^2\,\sigma^2\,\Delta^2 \\ & - & 32\,\dot{\phi}^2\,\sigma^4\,\Delta + 16\,\ddot{\phi}\,\sigma^4\,\Delta - 8\,\dot{\phi}\,\dot{\Delta}\,\sigma^4 \\ & - & 2\,e^{2\,\phi}\,r^2\,\dot{\phi}^2\,\sigma^4\,\Delta, \end{array} \tag{15}$$

$$\begin{array}{lcl} 0 & = & -8\,\dot{\Delta}^2\,\sigma^3\,\Delta - 8\,\Delta''\,\sigma^3\,\Delta^3 - 16\,\ddot{\sigma}\,\sigma^2\,\Delta^3 \\ & - & 8\,\dot{\Delta}\,\dot{\sigma}\,\sigma^2\,\Delta^2 + 4\,\sigma^5\,e^{2\,\phi}\,r^2\,\ddot{\phi}\,\Delta^2 + 8\,\Delta^4\,\Delta''\,\sigma \\ & - & 8\,\Delta^4\,e^{2\,\phi}\,r\,\phi'\,\sigma^3 - 24\,\Delta^4\,\Delta'\,\sigma' + 8\,\Delta'\,\sigma'\,\sigma^2\,\Delta^3 \\ & + & 32\,\dot{\sigma}^2\,\sigma\,\Delta^3 + 8\,\ddot{\Delta}\,\sigma^3\,\Delta^2 + 8\,\Delta'^2\,\sigma\,\Delta^3 \\ & - & 8\,\sigma^5\,\ddot{\Delta}\,\Delta + 16\,\sigma^5\,\dot{\Delta}^2 - 24\,\dot{\Delta}\,\dot{\sigma}\,\sigma^4\,\Delta \\ & + & 16\,\ddot{\sigma}\,\sigma^4\,\Delta^2 - 4\,\Delta^4\,e^{2\,\phi}\,r^2\,\phi''\,\sigma^3 \\ & - & 4\,e^{2\,\phi}\,\Delta'\,r^2\,\phi'\,\sigma^3\,\Delta^3 + 4\,\Delta^4\,e^{2\,\phi}\,r^2\,\phi'\,\sigma'\,\sigma^2 \\ & + & 4\,e^{2\,\phi}\,\dot{\sigma}\,r^2\,\dot{\phi}\,\sigma^4\,\Delta^2 - 4\,\sigma^5\,e^{2\,\phi}\,r^2\,\dot{\phi}\,\dot{\Delta}\,\Delta, \end{array} \tag{16}$$

$$\begin{array}{lcl} 0 & = & 2\,e^{2\,\phi}\,r\,\Delta''\,\Delta^3\,\sigma^3 + 128\,\dot{\phi}\,\dot{\sigma}\,\phi'\,\Delta^3\,\sigma^2 + 16\,\phi'\,\ddot{\Delta}\,\Delta^2\,\sigma^3 \\ & - & 32\,\Delta^4\,\phi'^2\,\Delta'\,\sigma + 32\,\dot{\phi}^2\,\Delta'\,\Delta^2\,\sigma^3 + 32\,\dot{\phi}'\,\dot{\Delta}\,\Delta^2\,\sigma^3 \\ & + & 16\,\dot{\phi}\,\Delta'\,\dot{\sigma}\,\Delta^2\,\sigma^2 - 64\,\dot{\phi}\,\dot{\Delta}\phi'\,\Delta^2\,\sigma^3 + 16\,\Delta^4\,\phi'\,\Delta''\,\sigma \\ & - & 16\,\ddot{\phi}\,\Delta'\,\Delta^2\,\sigma^3 - 16\,\phi'\,\dot{\Delta}^2\,\Delta\,\sigma^3 - 2\,e^{2\,\phi}\,r\,\Delta'\,\sigma'\,\Delta^3\,\sigma^2 \\ & - & 4\,\sigma^5\,e^{2\,\phi}\,r\,\dot{\Delta}^2 - 4\,e^{2\,\phi}\,r\,\ddot{\sigma}\,\Delta^2\,\sigma^4 - 48\,\Delta^4\,\phi'\,\Delta'\,\sigma' \\ & + & 32\,\ddot{\phi}\,\sigma'\,\Delta^3\,\sigma^2 + 16\,\Delta^4\,\phi''\,\Delta'\,\sigma - 16\,\dot{\phi}\dot{\Delta}\,\sigma'\,\Delta^2\,\sigma^2 \\ & - & 4\,\sigma^5\,e^{2\,\phi}\,r\,\dot{\phi}^2\,\Delta^2 + 64\,\phi'\,\dot{\sigma}^2\,\Delta^3\,\sigma + 4\,e^{2\,\phi}\,\Delta'\,\Delta^3\,\sigma^3 \\ & - & 64\,\dot{\phi}^2\,\sigma'\,\Delta^3\,\sigma^2 - 64\,\dot{\phi}'\,\dot{\sigma}\,\Delta^3\,\sigma^2 + 16\,\Delta'^2\,\phi'\,\Delta^3\,\sigma \\ & - & 32\,\phi'\,\ddot{\sigma}\,\Delta^3\,\sigma^2 - 16\,\phi'\,\dot{\Delta}\,\dot{\sigma}\,\Delta^2\,\sigma^2 \\ & + & 6\,e^{2\,\phi}\,\dot{\sigma}\,r\,\Delta'\,\Delta\,\sigma^4 + 2\,\sigma^5\,e^{2\,\phi}\,r\,\ddot{\Delta}\,\Delta - 4\,\Delta^4\,e^{2\,\phi}\,\sigma'\,\sigma^2 \\ & + & 4\,\Delta^4\,e^{2\,\phi}\,r\,\phi'^2\,\sigma^3, \end{array} \tag{17}$$

Nonzero coefficients for the field equations (14)-(17) with the variations (12):

$$\begin{array}{lcl} A_{4\,1} & = & -2\,\Delta'\,\sigma^3\,\Delta^2 + 4\,\sigma'\,\sigma^2\,\Delta^3, \\ A_{4\,3} & = & \Delta^4\,r\,\phi'^2\,\sigma^3\,e^{2\,\phi} + \Delta'\,\sigma^3\,e^{2\,\phi}\,\Delta^3 - \Delta^4\,\sigma'\,\sigma^2\,e^{2\,\phi} \\ & + & \dfrac{1}{2}\,r\,\Delta''\,\sigma^3\,e^{2\,\phi}\,\Delta^3 - \dfrac{1}{2}\,r\,\Delta'\,\sigma'\,\sigma^2\,e^{2\,\phi}\,\Delta^3, \\ A_{4\,4} & = & \Delta^4\,r\,\phi'\,\sigma^3\,e^{2\,\phi} - 6\,\Delta^4\,\Delta'\,\sigma' + 2\,\Delta^4\,\Delta''\,\sigma \\ & - & 8\,\Delta^4\,\phi'\,\Delta'\,\sigma + 2\,\Delta'^2\,\sigma\,\Delta^3, \\ A_{4\,5} & = & 2\,\Delta^4\,\Delta'\,\sigma, \\ B_{4\,1} & = & 2\,\phi'\,\sigma^3\,\Delta^2 + \dfrac{1}{4}\,\sigma^5\,r\,e^{2\,\phi}\,\Delta, \\ B_{4\,3} & = & -24\,\Delta^3\,\phi'\,\Delta'\,\sigma' + 8\,\Delta^3\,\phi''\,\Delta'\,\sigma \end{array}$$

$$
\begin{aligned}
&+ && 6\,\Delta'^2\,\phi'\,\sigma\,\Delta^2 - 16\,\Delta^3\,\phi'^2\,\Delta'\,\sigma \\
&+ && 2\,\Delta^3\,r\,\phi'^2\,\sigma^3\,e^{2\,\phi} - 2\,\Delta^3\,\sigma'\,\sigma^2\,e^{2\,\phi} \\
&- && \frac{3}{4}\,r\,\Delta'\,\sigma'\,\sigma^2\,e^{2\,\phi}\,\Delta^2 + \frac{3}{4}\,r\,\Delta''\,\sigma^3\,e^{2\,\phi}\,\Delta^2 \\
&+ && \frac{3}{2}\,\Delta'\,\sigma^3\,e^{2\,\phi}\,\Delta^2 + 8\,\Delta^3\,\phi'\,\Delta''\,\sigma, \\
B_{4\,4} &= && 2\,\Delta^4\,\phi''\,\sigma - 6\,\Delta^4\,\phi'\,\sigma' - \frac{1}{4}\,r\,\sigma'\,\sigma^2\,e^{2\,\phi}\,\Delta^3 \\
&- && 4\,\Delta^4\,\phi'^2\,\sigma + \frac{1}{2}\,\sigma^3\,e^{2\,\phi}\,\Delta^3 + 4\,\Delta'\,\phi'\,\sigma\,\Delta^3, \\
B_{4\,5} &= && 2\,\Delta^4\,\phi'\,\sigma + \frac{1}{4}\,r\,\sigma^3\,e^{2\,\phi}\,\Delta^3, \\
C_{4\,1} &= && -4\,\phi'\,\sigma^2\,\Delta^3 - \frac{1}{2}\,r\,\sigma^4\,e^{2\,\phi}\,\Delta^2, \\
C_{4\,3} &= && -\frac{1}{2}\,r\,\Delta'\,\sigma'\,\sigma\,e^{2\,\phi}\,\Delta^3 + \frac{3}{2}\,\Delta^4\,r\,\phi'^2\,\sigma^2\,e^{2\,\phi} \\
&+ && \frac{3}{2}\,\Delta'\,\sigma^2\,e^{2\,\phi}\,\Delta^3 - \Delta^4\,\sigma'\,\sigma\,e^{2\,\phi} \\
&+ && 2\,\Delta^4\,\phi'\,\Delta'' - 4\,\Delta^4\,\phi'^2\,\Delta' + 2\,\Delta'^2\,\phi'\,\Delta^3 \\
&+ && 2\,\Delta^4\,\phi''\,\Delta' + \frac{3}{4}\,r\,\Delta''\,\sigma^2\,e^{2\,\phi}\,\Delta^3, \\
C_{4\,4} &= && -\frac{1}{4}\,r\,\Delta'\,\sigma^2\,e^{2\,\phi}\,\Delta^3 - \frac{1}{2}\,\Delta^4\,\sigma^2\,e^{2\,\phi} - 6\,\Delta^4\,\phi'\,\Delta', \\
A_{3\,1} &= && \frac{1}{2}\,\sigma^5\,r^2\,e^{2\,\phi}\,\Delta^2, \\
A_{3\,3} &= && -\Delta^4\,r^2\,\phi''\,\sigma^3\,e^{2\,\phi} - 2\,\Delta^4\,r\,\phi'\,\sigma^3\,e^{2\,\phi} \\
&+ && \Delta^4\,r^2\,\phi'\,\sigma'\,\sigma^2\,e^{2\,\phi} - \Delta'\,r^2\,\phi'\,\sigma^3\,e^{2\,\phi}\,\Delta^3, \\
A_{3\,4} &= && -\Delta^4\,r\,\sigma^3\,e^{2\,\phi} + \frac{1}{2}\,\Delta^4\,r^2\,\sigma'\,\sigma^2\,e^{2\,\phi} \\
&- && \frac{1}{2}\,\Delta'\,r^2\,\sigma^3\,e^{2\,\phi}\,\Delta^3, \\
A_{3\,5} &= && -\frac{1}{2}\,\Delta^4\,r^2\,\sigma^3\,e^{2\,\phi}, \\
B_{3\,1} &= && \sigma^3\,\Delta^2 - \sigma^5\,\Delta, \\
B_{3\,3} &= && 3\,\Delta'^2\,\sigma\,\Delta^2 - 4\,\Delta^3\,r\,\phi'\,\sigma^3\,e^{2\,\phi} - 2\,\Delta^3\,r^2\,\phi''\,\sigma^3\,e^{2\,\phi} \\
&+ && 4\,\Delta^3\,\Delta''\,\sigma + 3\,\Delta'\,\sigma'\,\sigma^2\,\Delta^2 - 12\,\Delta^3\,\Delta'\,\sigma' \\
&- && 3\,\Delta''\,\sigma^3\,\Delta^2 - \frac{3}{2}\,\Delta'\,r^2\,\phi'\,\sigma^3\,e^{2\,\phi}\,\Delta^2 \\
&+ && 2\,\Delta^3\,r^2\,\phi'\,\sigma'\,\sigma^2\,e^{2\,\phi}, \\
B_{3\,4} &= && -3\,\Delta^4\,\sigma' - \frac{1}{2}\,r^2\,\phi'\,\sigma^3\,e^{2\,\phi}\,\Delta^3 + \sigma'\,\sigma^2\,\Delta^3 \\
&+ && 2\,\Delta'\,\sigma\,\Delta^3, \\
B_{3\,5} &= && \Delta^4\,\sigma - \sigma^3\,\Delta^3, \\
C_{3\,1} &= && -2\,\sigma^2\,\Delta^3 + 2\,\sigma^4\,\Delta^2, \\
C_{3\,3} &= && \Delta'^2\,\Delta^3 - 3\,\Delta^4\,r\,\phi'\,\sigma^2\,e^{2\,\phi} - \frac{3}{2}\,\Delta^4\,r^2\,\phi''\,\sigma^2\,e^{2\,\phi} \\
&+ && 2\,\Delta'\,\sigma'\,\sigma\,\Delta^3 - 3\,\Delta''\,\sigma^2\,\Delta^3 \\
&- && \frac{3}{2}\,\Delta'\,r^2\,\phi'\,\sigma^2\,e^{2\,\phi(r)}\,\Delta^3 + \Delta^4\,\Delta'' \\
&+ && \Delta^4\,r^2\,\phi'\,\sigma'\,\sigma\,e^{2\,\phi}, \\
C_{3\,4} &= && \Delta'\,\sigma^2\,\Delta^3 - 3\,\Delta^4\,\Delta' + \frac{1}{2}\,\Delta^4\,r^2\,\phi'\,\sigma^2\,e^{2\,\phi}, \\
A_{2\,1} &= && 2\,\sigma^4\,\Delta - 2\,\sigma^2\,\Delta^2, \\
A_{2\,3} &= && \frac{1}{2}\,r\,\Delta'\,\sigma^2\,e^{2\,\phi}\,\Delta^2 - \frac{1}{2}\,\sigma^4\,e^{2\,\phi}\,\Delta^2 \\
&+ && \frac{1}{2}\,\Delta^3\,\sigma^2\,e^{2\,\phi} - \frac{1}{2}\,\Delta^3\,r^2\,\phi'^2\,\sigma^2\,e^{2\,\phi}, \\
A_{2\,4} &= && -\frac{1}{2}\,\Delta^3\,r^2\,\phi'\,\sigma^2\,e^{2\,\phi} + 3\,\Delta^3\,\Delta' - \Delta'\,\sigma'^2\,\Delta'^2, \\
B_{2\,3} &= && -\frac{3}{4}\,\Delta^2\,r^2\,\phi'^2\,\sigma^2\,e^{2\,\phi} - \frac{1}{2}\,\sigma^4\,e^{2\,\phi}\,\Delta \\
&+ && \frac{1}{2}\,r\,\Delta'\,\sigma^2\,e^{2\,\phi}\,\Delta - 2\,\phi'\,\Delta'\,\sigma^2\,\Delta + \frac{3}{4}\,\Delta^2\,\sigma^2\,e^{2\,\phi} \\
&+ && 9\,\Delta^2\,\phi'\,\Delta', \\
B_{2\,4} &= && 3\,\Delta^3\,\phi' - \phi'\,\sigma^2\,\Delta^2 + \frac{1}{4}\,r\,\sigma^2\,e^{2\,\phi}\,\Delta^2, \\
C_{2\,3} &= && \frac{1}{2}\,\Delta^3\,\sigma\,e^{2\,\phi} - \frac{1}{2}\,\Delta^3\,r^2\,\phi'^2\,\sigma\,e^{2\,\phi} \\
&- && \sigma^3\,e^{2\,\phi}\,\Delta^2 + \frac{1}{2}\,r\,\Delta'\,\sigma'\,e^{2\,\phi}\,\Delta^2 - 2\,\phi'\,\Delta'\,\sigma\,\Delta^2, \\
A_{1\,1} &= && -\sigma^5 + \sigma^3\,\Delta, \\
A_{1\,3} &= && \frac{1}{2}\,r^2\,\phi'^2\,\sigma^3\,e^{2\,\phi}\,\Delta^2 - \frac{1}{2}\,r\,\sigma'\,\sigma^2\,e^{2\,\phi}\,\Delta^2, \\
A_{1\,4} &= && -3\,\Delta^3\,\sigma' + \sigma'\,\sigma^2\,\Delta^2 + \frac{1}{2}\,r^2\,\phi'\,\sigma^3\,e^{2\,\phi}\,\Delta^2 \\
&- && 4\,\Delta^3\,\phi'\,\sigma + 4\,\phi'\,\sigma^3\,\Delta^2, \\
A_{1\,5} &= && -\sigma^3\,\Delta^2 + \Delta^3\,\sigma, \\
B_{1\,3} &= && -\frac{1}{2}\,r\,\sigma'\,\sigma^2\,e^{2\,\phi}\,\Delta - 2\,\phi''\,\sigma^3\,\Delta \\
&+ && \frac{1}{2}\,r^2\,\phi'^2\,\sigma^3\,e^{2\,\phi}\,\Delta + 3\,\Delta^2\,\phi''\,\sigma \\
&- && 9\,\Delta^2\,\phi'\,\sigma' + 2\,\phi'\,\sigma'\,\sigma^2\,\Delta \\
&- && 6\,\Delta^2\,\phi'^2\,\sigma + 4\,\phi'^2\,\sigma^3\,\Delta, \\
C_{1\,3} &= && \Delta^3\,\phi'' - 2\,\Delta^3\,\phi'^2 - 3\,\phi''\,\sigma^2\,\Delta^2 \\
&+ && 2\,\phi'\,\sigma'\,\sigma\,\Delta^2 - \frac{1}{2}\,r\,\sigma'\,\sigma\,e^{2\,\phi}\,\Delta^2 \\
&+ && \frac{3}{4}\,\Delta^2\,r^2\,\phi'^2\,\sigma^2\,e^{2\,\phi} + 6\,\phi'^2\,\sigma^2\,\Delta^2, \\
C_{1\,4} &= && \phi'\,\sigma^2\,\Delta^2 - \frac{1}{4}\,r\,\sigma^2\,e^{2\,\phi}\,\Delta^2 - 3\,\Delta^3\,\phi'.
\end{aligned}
$$

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